5. Functional Equations

Notations: $\mathbb{Z}$ denotes integers; $\mathbb{Z}^+$ or $\mathbb{N}$ denotes positive integers; $\mathbb{N}_0$ denotes nonnegative integers; $\mathbb{Q}$ denotes rational numbers; $\mathbb{R}$ denotes real numbers; $\mathbb{R}^+$ denotes positive real numbers; $\mathbb{C}$ denotes complex numbers.

In simple cases, a functional equation can be solved by introducing some substitutions to yield more informations or additional equations.

Example. (1) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1-x) = 2x - x^4$$

for all $x \in \mathbb{R}$.

Solution. Replacing x by $1-x$, we have $(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4$. Since $f(1-x) = 2x - x^4 - x^2 f(x)$ by the given equation, we have $(1-x)^2(2x - x^4 - x^2 f(x)) + f(x) = 2(1-x) - (1-x)^4$. Solving for $f(x)$, we have

$$f(x) = \frac{2(1-x) - (1-x)^4 - (1-x)^2(2x - x^4)}{1 - x^2(1-x)^2} = 1 - x^2.$$

Check: For $f(x) = 1 - x^2$, $x^2 f(x) + f(1-x) = x^2(1-x^2) + (1-(1-x)^2) = 2x - x^4$.

For certain type of functional equations, a standard approach to solving the problem is to determine some special values (such as $f(0)$ or $f(1)$), then inductively determine $f(n)$ for $n \in \mathbb{N}_0$, follow by reciprocal values $f(\frac{1}{n})$ and use density to find all $f(x)$, $x \in \mathbb{R}$. The following are examples of such approach.

Example. (2) Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x+y) = f(x) + f(y) \qquad \text{(Cauchy Equation)}$$

for all $x, y \in \mathbb{Q}$.

Solution. Step 1 Taking $x = 0 = y$, we get $f(0) = f(0) + f(0) \Rightarrow f(0) = 0$.

Step 2 We will prove $f(kx) = kf(x)$ for $k \in \mathbb{N}$, $x \in \mathbb{Q}$ by induction. This is true for $k = 1$. Assume this is true for k. Taking $y = kx$, we get

$$f(x + kx) = f(x) + f(kx) = f(x) + kf(x) = (k+1) f(x).$$

Step 3 Taking $y = -x$, we get $0 = f(0) = f(x + (-x)) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$. So $f(-kx) = -f(kx) = -kf(x)$ for $k \in \mathbb{N}$. Therefore, $f(kx) = kf(x)$ for $k \in \mathbb{Z}$, $x \in \mathbb{Q}$.

Step 4 Taking $x = \frac{1}{k}$, we get $f(1) = f(k\frac{1}{k}) = kf(\frac{1}{k}) \Rightarrow f(\frac{1}{k}) = \frac{1}{k} f(1)$.

Step 5 For $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $f(\frac{m}{n}) = mf(\frac{1}{n}) = \frac{m}{n} f(1)$. Therefore, $f(x) = cx$ with $c(= f(1)) \in \mathbb{Q}$.

Check: For $f(x) = cx$ with $c \in \mathbb{Q}$, $f(x+y) = c(x+y) = cx + cy = f(x) + f(y)$.

In dealing with functions on $\mathbb{R}$, after finding the function on $\mathbb{Q}$, we can often finish the problem by using the following fact. (It follows from decimal representation of real numbers. For example, $\pi = 3.14159\ldots$ is the limits of $3, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10000}, \frac{314159}{100000}, \ldots$ and also $4, \frac{32}{10}, \frac{315}{100}, \frac{3142}{1000}, \ldots$.)

Density of Rational Numbers. *For every real number x, there are rational numbers $p_1, p_2, \ldots$ increase to x and there are rational numbers $q_1, q_2, \ldots$ decrease to x. We denote this by $p_n \nearrow x$ and $q_n \searrow x$ as $n \to +\infty$.*

Example. (3) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) > 0$ for $x > 0$.

Solution. Step 1 By example 2, $f(x) = xf(1)$ for $x \in \mathbb{Q}$.

Step 2 If $x > y$, then $x - y > 0$. So

$$f(x) = f((x-y) + y) = f(x-y) + f(y) > f(y).$$

So, f is strictly increasing.

Step 3 If $x \in \mathbb{R}$, then by the density of rational numbers, there are $p_n < x < q_n$ such that $p_n \nearrow x$ and $q_n \searrow x$ as $n \to +\infty$. So by step 2, $p_n f(1) = f(p_n) < f(x) < f(q_n) = q_n f(1)$. As $n \to +\infty$, $p_n f(1) \nearrow xf(1)$ and $q_n f(1) \searrow xf(1)$. So $p_n f(1)$ and $q_n f(1)$ will squeeze $f(x)$ to $xf(1)$. We get $f(x) = xf(1)$ for all $x \in \mathbb{R}$. Therefore, $f(x) = cx$ with $c(= f(1)) > 0$.

Check: For $f(x) = cx$ with $c > 0$, $f(x+y) = c(x+y) = cx + cy = f(x) + f(y)$ and $f(x) = cx > 0$ for $x > 0$.

The concept of a fixed point is another useful idea in attacking a functional equations. Knowing all the fixed points are important in certain types of functional equations.

Definitions. w is a *fixed point* of a function f if $f(w) = w$. Let $f^{(1)} = f$ and $f^{(n)} = f \circ f^{(n-1)}$ for $n = 2, 3, 4, \ldots$, the function $f^{(n)}$ is called the *n-th iterate of f*.

Let S_n be the set of fixed points of $f^{(n)}$. Observe that if x is a fixed point of $f^{(n)}$, then $f(x)$ is also a fixed point of $f^{(n)}$ because $f^{(n)}(f(x)) = f^{(n+1)}(x) = f(f^{(n)}(x)) = f(x)$. So f takes S_n to itself. Also f is injective on S_n because if $f(a) = f(b)$ for $a, b \in S_n$, then $a = f^{(n)}(a) = f^{(n-1)}(f(a)) = f^{(n-1)}(f(b)) = f^{(n)}(b) = b$. This means that if S_n is a finite set, then f is a permutation of S_n.

Since $g(x) = x$ implies $g^{(2)}(x) = g(g((x)) = g(x) = x$, so the fixed points of g are also fixed points of $g^{(2)}$. Letting $g = f, f^{(2)}, f^{(4)}, f^{(8)}, \ldots$, respectively, we get $S_1 \subseteq S_2 \subseteq S_4 \subseteq S_8 \subseteq \cdots$.

Example. (4) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2$ for all $x \in \mathbb{R}$.

Solution. Assume such f exists. It turns out S_2 and S_4 are useful for this problem. The fixed points of $f^{(2)}$ are the roots of $x = x^2 - 2$, i.e. $S_2 = \{-1, 2\}$. The fixed points of $f^{(4)}$ are the roots of $x = x^4 - 4x^2 + 2$. i.e. $S_4 = \{-1, 2, \frac{-1 \pm \sqrt{5}}{2}\}$. Let $c = \frac{-1 + \sqrt{5}}{2}, d = \frac{-1 - \sqrt{5}}{2}$. Since f permutes S_2 and $c, d \in S_4 \setminus S_2$, $f(c) = c$ or d. If $f(c) = c$, then $f^{(2)}(c) = c$ implies c is a fixed point of $f^{(2)}$, which is not true. So $f(c) = d$ and hence $f(d) = c$. Then $c = f(d) = f(f(c)) = f^{(2)}(c)$, again a contradiction. So no such f can exist.

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(5) (1983 IMO) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(xf(y)) = yf(x)$ for all $x, y \in \mathbb{R}^+$ and as $x \to +\infty$, $f(x) \to 0$.

Solution. Step 1 Taking $x = 1 = y$, we get $f(f(1)) = f(1)$. Taking $x = 1, y = f(1)$, we get $f(f(f(1))) = f(1)^2$. Then

$$f(1)^2 = f(f(f(1))) = f(f(1)) = f(1) \Rightarrow f(1) = 1$$

since $f(1) \in \mathbb{R}^+$. So 1 is a fixed point of f.

Step 2 Taking $y = x$, we get $f(xf(x)) = xf(x)$. So $w = xf(x)$ is a fixed point of f for every $x \in \mathbb{R}^+$.

Step 3 Suppose f has a fixed point $x > 1$. By step 2, $xf(x) = x^2$ is also a fixed point, $x^2 f(x^2) = x^4$ is also a fixed point, ... So x^{2^n}'s are fixed points. Since $x > 1$, $x^{2^n} \to +\infty$, but $f(x^{2^n}) = x^{2^n} \to +\infty$, not 0. This contradicts $f(x) \to 0$ as $x \to +\infty$. So f does not have any fixed point $x > 1$.

Step 4 Suppose f has a fixed point $x \in (0, 1)$. Then

$$1 = f(\frac{1}{x}x) = f(\frac{1}{x}f(x)) = xf(\frac{1}{x}) \Rightarrow f(\frac{1}{x}) = \frac{1}{x},$$

i.e. f has a fixed point $\frac{1}{x} > 1$, contradicting step 3. So f does not have any fixed point $x \in (0, 1)$.

Step 5 Steps 1, 3, 4 showed the only fixed point of f is 1. By step 2, we get $xf(x) = 1 \Rightarrow f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^+$.

Check: For $f(x) = \frac{1}{x}$, $f(xf(y)) = f(\frac{x}{y}) = \frac{y}{x} = yf(x)$. As $x \to +\infty$, $f(x) = \frac{1}{x} \to 0$.

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(6) (1996 IMO) Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(m + f(n)) = f(f(m)) + f(n)$ for all $m, n \in \mathbb{N}_0$.

Solution. Step 1 Taking $m = 0 = n$, we get $f(f(0)) = f(f(0)) + f(0) \Rightarrow f(0) = 0$. Taking $m = 0$, we get $f(f(n)) = f(n)$, i.e. $f(n)$ is a fixed point of f for every $n \in \mathbb{N}_0$. Also the equation becomes $f(m + f(n)) = f(m) + f(n)$.

Step 2 If w is a fixed point of f, then we will show kw is a fixed point of f for all $k \in \mathbb{N}_0$. The cases $k = 0, 1$ are known. Suppose kw is a fixed point, then $f(kw + w) = f(kw + f(w)) = f(kw) + f(w) = kw + w$ and so $(k + 1)w$ is also a fixed point.

Step 3 If 0 is the only fixed point of f, then $f(n) = 0$ for all $n \in \mathbb{N}_0$ by step 1. Obviously, the zero function is a solution.

Otherwise, f has a least fixed point $w > 0$. We will show the only fixed points are kw, $k \in \mathbb{N}_0$. Suppose x is a fixed point. By the division algorithm, $x = kw + r$, where $0 \le r < w$. We have

$$x = f(x) = f(r + kw) = f(r + f(kw)) = f(r) + f(kw) = f(r) + kw,$$

So $f(r) = x - kw = r$. Since w is the least positive fixed point, $r = 0$ and $x = kw$.

Since $f(n)$ is a fixed point for all $n \in \mathbb{N}_0$ by step 1, $f(n) = c_n w$ for some $c_n \in \mathbb{N}_0$. We have $c_0 = 0$.

<u>Step 4</u> For $n \in \mathbb{N}_0$, by the division algorithm, $n = kw + r$, $0 \le r < w$. We have

$$\begin{aligned} f(n) &= f(r + kw) = f(r + f(kw)) = f(r) + f(kw) \\ &= c_r w + kw = (c_r + k)w = (c_r + \left[\frac{n}{w}\right])w. \end{aligned}$$

Check: For each $w > 0$, let $c_0 = 0$ and let $c_1, \ldots, c_{w-1} \in \mathbb{N}_0$ be arbitrary. The functions $f(n) = (c_r + [\frac{n}{w}])w$, where r is the remainder of n divided by w, (and the zero function) are all the solutions. Write $m = kw + r, n = lw + s$ with $0 \le r, s < w$. Then

$$f(m + f(n)) = f(r + kw + (c_s + l)w) = c_r w + kw + c_s w + lw = f(f(m)) + f(n).$$

The above examples showed traditional or systematical ways of solving functional equations. The following examples show some other approaches to deal with these equations.

Example. (7) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(m) + f(n)) = m + n$ for all $m, n \in \mathbb{N}$.

Solution. Clearly, the identity function $f(x) = x$ is a solution. We will show that is the only solution.

To show $f(1) = 1$, suppose $f(1) = t > 1$. Let $s = f(t - 1) > 0$. Observe that if $f(m) = n$, then $f(2n) = f(f(m) + f(m)) = 2m$. So $f(2t) = 2$ and $f(2s) = 2t - 2$. Then $2s + 2t = f(f(2s) + f(2t)) = f(2t) = 2 \Rightarrow t < 1$, a contradiction. Therefore, $f(1) = 1$.

Inductively, suppose $f(n) = n$. Then $f(n + 1) = f(f(n) + f(1)) = n + 1$. Therefore, $f(n) = n$ for all $n \in \mathbb{N}$ by mathematical induction.

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(8) (1987 IMO) Prove that there is no function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(f(n)) = n + 1987$.

Solution. Suppose there is such a function f. Then f is injective, i.e. $f(a) = f(b) \Rightarrow a + 1987 = f(f(a)) = f(f(b)) = b + 1987 \Rightarrow a = b$.

Suppose $f(n)$ misses exactly k distinct values $c_1, \ldots, c_k$ in $\mathbb{N}_0$, i.e. $f(n) \ne c_1, \ldots, c_k$ for all $n \in \mathbb{N}_0$. Then $f(f(n))$ misses the $2k$ distinct values $c_1, \ldots, c_k$ and $f(c_1), \ldots, f(c_k)$ in $\mathbb{N}_0$. (The $f(c_j)$'s are distinct because f is injective.) Now if $w \ne c_1, \ldots, c_k, f(c_1), \ldots, f(c_k)$, then there is $m \in \mathbb{N}_0$ such that $f(m) = w$. Since $w \ne f(c_j)$, $m \ne c_j$, so there is $n \in \mathbb{N}_0$ such that $f(n) = m$, then $f(f(n)) = w$. This shows $f(f(n))$ misses only the $2k$ values $c_1, \ldots, c_k$, $f(c_1), \ldots, f(c_k)$ and no others. Since $n + 1987$ misses the 1987 values $0, 1, \ldots, 1986$ and $2k \ne 1987$, this is a contradiction.

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(9) (1999 IMO) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Solution. Let $c = f(0)$. Setting $x = y = 0$, we get $f(-c) = f(c) + c - 1$. So $c \ne 0$. Let A be the range of f, then for $x = f(y) \in A$, we have $c = f(0) = f(x) + x^2 + f(x) - 1$. Solving for $f(x)$, this gives $f(x) = \dfrac{c+1}{2} - \dfrac{x^2}{2}$.

Next, if we set $y = 0$, we get

$$\{f(x - c) - f(x) : x \in \mathbb{R}\} = \{cx + f(c) - 1 : x \in \mathbb{R}\} = \mathbb{R}$$

because $c \ne 0$. This means $A - A = \{y_1 - y_2 : y_1, y_2 \in A\} = \mathbb{R}$.

Now for an arbitrary $x \in \mathbb{R}$, let $y_1, y_2 \in A$ be such that $y_1 - y_2 = x$. Then

$$\begin{aligned} f(x) &= f(y_1 - y_2) = f(y_2) + y_1 y_2 + f(y_1) - 1 \\ &= \frac{c+1}{2} - \frac{y_2^2}{2} + y_1 y_2 + \frac{c+1}{2} - \frac{y_1^2}{2} - 1 \\ &= c - \frac{(y_1 - y_2)^2}{2} = c - \frac{x^2}{2}. \end{aligned}$$

However, for $x \in A$, $f(x) = \dfrac{c+1}{2} - \dfrac{x^2}{2}$. So $c = 1$. Therefore, $f(x) = 1 - \dfrac{x^2}{2}$ for all $x \in \mathbb{R}$.

Exercises

1. Find all functions $f : \mathbb{N}_0 \to \mathbb{Q}$ such that $f(1) \neq 0$ and

$$f(x+y^2) = f(x) + 2(f(y))^2 \quad \text{for all } x, y \in \mathbb{N}_0.$$

2. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(1) = 2$ and

$$f(xy) = f(x)f(y) - f(x+y) + 1 \quad \text{for all } x, y \in \mathbb{Q}.$$

3. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that

$$f(x)f(y) = f(x+y) \quad \text{for all } x, y \in \mathbb{Q}.$$

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
   (a) $f(x+y) = f(x) + f(y) + 2xy$ for all $x, y \in \mathbb{R}$ and
   (b) $\lim_{x\to 0} \frac{f(x)}{x} = 1$.

   (*Hint*: For $n \in \mathbb{N}$, consider $y = x,\ y = 2x,\ \ldots,\ y = (n-1)x$.)

5. (1986 IMO) Find all functions $f : [0, \infty) \to [0, \infty)$ such that
   (a) $f(xf(y))f(y) = f(x+y)$ for $x, y \geq 0$ and
   (b) $f(2) = 0$ and $f(x) \neq 0$ for $0 \leq x < 2$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}^+$ is such that

$$f(\sqrt{x^2+y^2}) = f(x)f(y) \quad \text{for every } x, y \in \mathbb{R}.$$

   Find $f(x)$ for $x \in \mathbb{Q}$ in terms of $f(1)$.

*7. (1990 IMO) Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y} \quad \text{for all } x, y \in \mathbb{Q}^+.$$

*8. (1994 IMO) Let S be the set of real numbers greater than -1. Find all functions $f : S \to S$ such that
   (a) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$ and
   (b) $\frac{f(x)}{x}$ is strictly increasing for $-1 < x < 0$ and for $0 < x$.

   (*Hint*: Show f can only have a fixed point at 0.)

*9. (1992 IMO) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = y + (f(x))^2 \quad \text{for all } x, y \in \mathbb{R}.$$

   (*Hint*: Assume $f(0) = 0$, then show $x > 0 \Rightarrow f(x) > 0$, and f is increasing.)

*10. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(2) = 2$ and

$$f\left(\frac{x+y}{x-y}\right) = \frac{f(x)+f(y)}{f(x)-f(y)} \quad \text{for } x \neq y.$$

   (*Hint*: Try $y = cx$ for different $c \in \mathbb{Q}$ and $y = x - 2$.)